

# SOME PROPERTIES OF LUBIN-TATE COHOMOLOGY FOR CLASSIFYING SPACES OF FINITE GROUPS

ANDREW BAKER AND BIRGIT RICHTER

**ABSTRACT.** We consider brave new cochain extensions  $F(BG_+, R) \rightarrow F(EG_+, R)$ , where  $R$  is either a Lubin-Tate spectrum  $E_n$  or the related 2-periodic Morava K-theory  $K_n$ , and  $G$  is a finite group. When  $R$  is an Eilenberg-Mac Lane spectrum, in some good cases such an extension is a  $G$ -Galois extension in the sense of John Rognes, but not always faithful. We prove that for  $E_n$  and  $K_n$  these extensions are always faithful in the  $K_n$  local category. However, for a cyclic  $p$ -group  $C_{p^r}$ , the cochain extension  $F(BC_{p^r+}, E_n) \rightarrow F(EC_{p^r+}, E_n)$  is not a Galois extension because it ramifies. As a consequence, it follows that the  $E_n$ -theory Eilenberg-Moore spectral sequence for  $G$  and  $BG$  does not always converge to its expected target.

## 1. INTRODUCTION

In the algebraic Galois theory of commutative rings [6], faithful flatness is a property implied by separability. However, in the topological analogue, the brave new Galois theory of Rognes [19], this is not true. The simplest counterexample, due to Ben Wieland [20], is provided by the  $C_2$ -Galois extension

$$(1.1) \quad F(BC_{2+}, H\mathbb{F}_2) \rightarrow F(EC_{2+}, H\mathbb{F}_2) \sim H\mathbb{F}_2$$

which is not faithful. This example relies on the algebraic fact that

$$\pi_*(F(BC_{2+}, H\mathbb{F}_2)) = H^{-*}(BC_2; \mathbb{F}_2)$$

is a polynomial algebra and so has finite global dimension.

In this note we consider this question for a Lubin-Tate spectrum  $E_n$  and the related Morava K-theory  $K_n$ , and show that for any finite group  $G$ , the extension

$$(1.2) \quad E_n^{BG} = F(BG_+, E_n) \rightarrow F(EG_+, E_n) \sim E_n$$

is faithful as an  $E_n$ -module. We also show that the non-commutative extension

$$(1.3) \quad F(BG_+, K_n) \rightarrow F(EG_+, K_n) \sim K_n$$

is faithful and  $F(BG_+, K_n)$  is a faithful  $E_n$ -module. A crucial difference from  $F(BG_+, H\mathbb{F}_p)$  is that  $K_n^*(BG_+)$  is always an Artinian algebra over  $(K_n)_*$ , and so if  $K_n^*(BG_+) \neq K_n^*$  then it has infinite global dimension by Proposition 2.2.

Our approach to this involves introducing an analogue of the algebraic socle series for a module over an Artinian ring, and we show that this behaves well enough to prove our result.

We show in Section 5 that for a cyclic  $p$ -group  $C_{p^r}$ , the cochain extension  $F(BC_{p^r+}, E_n) \rightarrow F(EC_{p^r+}, E_n)$  is ramified and hence it is not a Galois extension. As a consequence it follows that the  $E_n$ -theory Eilenberg-Moore spectral sequence for such groups does not converge to its expected target, whereas work of Tilman Bauer indicates that this is not the case for Morava K-theory.

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**Notation, etc.** In discussing purely algebraic notions we will often use boldface symbols  $\mathbf{A}, \mathbf{M}, \dots$  to denote rings, modules, etc, while for topological objects such as  $S$ -algebras and their modules we will use italic symbols  $A, M, \dots$ , thereby hopefully reducing the possibility of confusion between the two settings. For an associative  $S$ -algebra  $A$ , we denote by  $\mathcal{D}_A$  the derived category of  $A$ -module spectra defined in [7, chapter III, construction 2.11].

We follow Lam [12, theorem 19.1] in using the phrase *local ring* to indicate a ring with a unique maximal left ideal (necessarily 2-sided and equal to its Jacobson radical); the quotient of such a ring by its Jacobson radical is a division ring. For non-commutative rings other terminology is often encountered such as *scalar local ring*.

**Brave new Galois extensions.** The following definition of a Galois extension is due to John Rognes [19]. Let  $A$  be a commutative  $S$ -algebra and let  $B$  be a commutative cofibrant  $A$ -algebra. Let  $G$  be a finite (discrete) group and suppose that there is an action of  $G$  on  $B$  by commutative  $A$ -algebra morphisms. Then  $B/A$  is a  *$G$ -Galois extension* if it satisfies the following two conditions:

- The natural map

$$A \longrightarrow B^{hG} = F(EG_+, B)^G$$

is a weak equivalence of  $A$ -algebras.

- There is a natural equivalence of  $B$ -algebras

$$\Theta: B \wedge_A B \xrightarrow{\sim} F(G_+, B)$$

induced from the action of  $G$  on the right hand factor of  $B$ .

Furthermore,  $B/A$  is a *faithful  $G$ -Galois extension* if it also satisfies

- $B$  is faithful as an  $A$ -module, i.e., for any  $A$ -module  $M$ ,  $B \wedge_A M \sim *$  implies that  $M \sim *$ .

Examples like (1.1) show that not every Galois extension is faithful.

## 2. RECOLLECTIONS ON MODULES OVER ARTINIAN ALGEBRAS

In this section we review some standard algebraic background material; good sources for this are [1, 12].

Let  $\mathbf{D}$  be a division ring. A ring  $\mathbf{A}$  equipped with homomorphisms of rings  $\eta: \mathbf{D} \longrightarrow \mathbf{A}$  and  $\varepsilon: \mathbf{A} \longrightarrow \mathbf{D}$  is an *augmented  $\mathbf{D}$ -algebra* if the following diagram commutes.

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{=} & \mathbf{D} \\ & \searrow \eta & \nearrow \varepsilon \\ & \mathbf{A} & \end{array}$$

The augmentation  $\varepsilon$  splits the unit  $\eta$ . We will also say that  $\mathbf{A}$  is an *Artinian local  $\mathbf{D}$ -algebra* if it is Artinian and local.

If  $\mathbf{A}$  is an Artinian local augmented  $\mathbf{D}$ -algebra, then the Jacobson radical of  $\mathbf{A}$  is

$$\mathbf{J} = \text{rad}(\mathbf{A}) = \ker \varepsilon.$$

By [12, theorem 4.12],  $\mathbf{J}$  is nilpotent, say  $\mathbf{J}^e = 0$  and  $\mathbf{J}^{e-1} \neq 0$ .

**Lemma 2.1.** *Let  $\mathbf{A}$  be as above and let  $\mathbf{M}$  be a left  $\mathbf{A}$ -module. If  $\mathbf{D} \otimes_{\mathbf{A}} \mathbf{M} = 0$ , then  $\mathbf{M} = 0$ .*

*Proof.* Comparing the two horizontal exact sequences

$$\begin{array}{ccccccc} \mathbf{J} \otimes_{\mathbf{A}} \mathbf{M} & \longrightarrow & \mathbf{A} \otimes_{\mathbf{A}} \mathbf{M} & \longrightarrow & \mathbf{D} \otimes_{\mathbf{A}} \mathbf{M} & \longrightarrow & 0 \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbf{J}\mathbf{M} & \longrightarrow & \mathbf{M} & \longrightarrow & \mathbf{M}/\mathbf{J}\mathbf{M} \longrightarrow 0 \end{array}$$

we see that if  $\mathbf{D} \otimes_{\mathbf{A}} \mathbf{M} = 0$  then

$$\mathbf{M} = \mathbf{J}\mathbf{M} = \dots = \mathbf{J}^e \mathbf{M} = 0.$$

□

Let  $M$  be a left  $A$ -module. The *socle* of  $M$  is the submodule

$$\text{soc}^1 M = \text{soc } M = \{x \in M : Jx = 0\},$$

which can also be characterized as the sum of all the simple  $A$ -submodules of  $M$ . The *socle series* of  $M$  is the increasing sequence of submodules

$$0 = \text{soc}^0 M \subseteq \text{soc}^1 M \subseteq \dots \subseteq \text{soc}^k M \subseteq \text{soc}^{k+1} M \subseteq \dots \subseteq M,$$

where for each  $k$  the following is a pullback square

$$\begin{array}{ccc} \text{soc}^{k+1} M & \longrightarrow & \text{soc}(M / \text{soc}^k M) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M / \text{soc}^k M \end{array}$$

so we have

$$\text{soc}^k M = \{x \in M : J^k x = 0\},$$

and

$$\text{soc}^e M = M.$$

In fact, for small  $k$

$$\text{soc}^k M \subset \text{soc}^{k+1} M,$$

until we reach a value  $k = k_0 \leq e$  for which  $\text{soc}^{k_0} M = M$ .

It is also clear that given a homomorphism  $\varphi: M \rightarrow N$  of  $A$ -modules there are compatible homomorphisms

$$\text{soc}^k M \rightarrow \text{soc}^k N.$$

For details on the socle series see [12], especially Ex. 4.18, and [1, chapter I, section 1].

We end this section with a result that supplies an algebraic backdrop for some of our later work. We give a proof suggested by K. Brown.

**Proposition 2.2.** *Let  $A$  be a local left-Artinian ring which is not a division ring. Then*

$$\text{proj dim}(A / \text{rad}(A)) = \text{gl dim } A = \infty,$$

*where  $A / \text{rad}(A)$  is the unique simple left  $A$ -module.*

*Proof.* Since  $A$  is local, it has only one simple module and therefore

$$\text{proj dim}(A / \text{rad}(A)) = \text{gl dim } A.$$

Also, since  $A$  is Artinian it has a left ideal  $I$  isomorphic to  $A / \text{rad}(A)$ . The corresponding exact sequence

$$(2.1) \quad 0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

cannot split since  $A$  is local and therefore it has no non-trivial idempotents.

If

$$\text{proj dim}(A / \text{rad}(A)) = \text{gl dim } A < \infty,$$

then (2.1) would give

$$\text{proj dim}(A / \text{rad}(A)) + 1 = \text{proj dim}(A/I) \leq \text{gl dim } A = \text{proj dim}(A / \text{rad}(A)),$$

which is impossible.  $\square$

**Remark 2.3.** We end this section by noting that the above discussion works as well if we assume that  $A$  is graded, provided this is suitably interpreted. In our work below we are interested in  $\mathbb{Z}$ -gradings which are also 2-periodic, *i.e.*, for all  $n \in \mathbb{Z}$ ,  $(-)_{n+2} = (-)_n$ . This can be interpreted as a  $\mathbb{Z}/2$ -grading.

### 3. SOCLE SERIES IN TOPOLOGY

Let  $D$  be an  $S$ -algebra for which  $\pi_0 D$  is a non-trivial division ring,  $\pi_1 D = 0$ , and the graded ring  $\pi_* D = \mathbf{D}$  has period two. Suppose that  $A$  is an  $S$ -algebra both under and over  $D$ , giving the following diagram of morphisms of  $S$ -algebras.

$$(3.1) \quad \begin{array}{ccc} D & \xrightarrow{=} & D \\ & \searrow \eta & \nearrow \varepsilon \\ & A & \end{array}$$

We assume that  $\mathbf{A} = \pi_* A$  is an Artinian local augmented  $\mathbf{D}$ -algebra, so that the augmentation ideal  $\ker \varepsilon$  is the Jacobson radical of  $\mathbf{A}$ ,  $\text{rad}(\mathbf{A})$ , and also  $\text{rad}(\mathbf{A})^e = 0$  and  $\text{rad}(\mathbf{A})^{e-1} \neq 0$ .

**Remark 3.1.** Let  $M$  be a left  $A$ -module. Then  $\mathbf{M} = \pi_* M$  is a left  $\mathbf{A}$ -module and its socle  $\text{soc } \mathbf{M}$  is a  $\mathbf{D}$ -module through both the unit  $\eta$  and the augmentation  $\varepsilon$ , and these module structures agree since  $\text{rad}(\mathbf{A}) = \ker \varepsilon$ .

**Theorem 3.2.** *There are functors  $\text{soc}^k: \mathcal{D}_A \rightarrow \mathcal{D}_A$  for  $0 \leq k \leq e$  such that*

- (a) *for each  $k$ ,  $\pi_*(\text{soc}^k M) = \text{soc}^k \mathbf{M}$ ;*
- (b) *there are natural transformations  $\text{soc}^k M \rightarrow \text{soc}^{k+1} M$  giving a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_* \text{soc}^1 M & \longrightarrow & \pi_* \text{soc}^2 M & \longrightarrow & \cdots \longrightarrow \pi_* \text{soc}^e M \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{soc}^1 \mathbf{M} & \longrightarrow & \text{soc}^2 \mathbf{M} & \longrightarrow & \cdots \longrightarrow \text{soc}^e \mathbf{M} \longrightarrow 0 \end{array}$$

*which is natural with respect to morphisms of  $A$ -modules.*

*Proof.* As  $\mathbf{D}$  is a graded division ring,  $\text{soc } \mathbf{M}$  is a  $\mathbf{D}$ -vector space. Since  $M$  is a  $D$ -module via the unit we can find a morphism of  $D$ -modules

$$(3.2) \quad \bigvee_j \Sigma^{s(j)} D \longrightarrow M$$

to realize an algebraic isomorphism

$$\bigoplus_j D_{*-s(j)} \xrightarrow{\cong} \text{soc } \mathbf{M} \subseteq \mathbf{M}.$$

Now Remark 3.1 implies that the morphism of (3.2) is actually one of  $A$ -modules. We set  $\text{soc } M = \bigvee_j \Sigma^{s(j)} D$ .

Now we can repeat this on the cofibre  $M/\text{soc } M$  of the map  $\text{soc } M \rightarrow M$ , obtaining  $\text{soc}(M/\text{soc } M) \rightarrow M/\text{soc } M$ . We then define  $\text{soc}^2 M$  using the right hand pullback square in the diagram

$$\begin{array}{ccccc} \text{soc } M & \longrightarrow & \text{soc}^2 M & \longrightarrow & \text{soc}(M/\text{soc } M) \\ \downarrow = & & \downarrow & & \downarrow \\ \text{soc } M & \longrightarrow & M & \longrightarrow & M/\text{soc } M \end{array}$$

from which we see by a standard diagram chase that  $\pi_*(\text{soc}^2 M) \cong \text{soc}^2 \mathbf{M}$ . Continuing in this way we inductively build the socle tower

$$* \rightarrow \text{soc}^1 M \rightarrow \text{soc}^2 M \rightarrow \cdots \rightarrow \text{soc}^{e-1} M \rightarrow \text{soc}^e M = M,$$

using pullback squares

$$\begin{array}{ccc} \text{soc}^{k+1} M & \longrightarrow & \text{soc}(M/\text{soc}^k M) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/\text{soc}^k M \end{array}$$

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for each  $k$ . These satisfy

$$\pi_*(\text{soc}^k M) = \text{soc}^k \mathbf{M}. \quad \square$$

An important consequence of this construction is that there is a minimal  $k_0$  for which  $\text{soc}^{k_0} M = M$ , so since  $\text{soc}^{k_0-1} \mathbf{M} \neq \mathbf{M}$ , using the fibre sequence

$$(3.3) \quad \text{soc}^{k_0-1} M \longrightarrow M \longrightarrow M / \text{soc}^{k_0-1} M,$$

we obtain  $\pi_*(M / \text{soc}^{k_0-1} M) \neq 0$ .

**Lemma 3.3.** *The  $A$ -module  $D$  satisfies  $\pi_*(D \wedge_A D) \neq 0$ .*

*Proof.* There is a diagram of left  $D$ -modules induced from (3.1)

$$\begin{array}{ccc} D \wedge_D D & \xrightarrow{=} & D \wedge_D D \\ & \searrow & \nearrow \\ & D \wedge_A D & \end{array}$$

in which  $D \wedge_D D \cong D$ . On applying  $\pi_*(-)$  we see that  $\pi_*(D \wedge_A D) \neq 0$ .  $\square$

**Theorem 3.4.** *Let  $M$  be an  $A$ -module for which  $\pi_* M \neq 0$ . Then  $\pi_*(D \wedge_A M) \neq 0$ , i.e.,  $D$  is a faithful  $A$ -module.*

*Proof.* Using the socle series we can find a fibration sequence as in (3.3),

$$(3.4) \quad M' \longrightarrow M \longrightarrow M'',$$

where  $\mathbf{M}'' = \pi_* M'' \neq 0$ ,  $J\mathbf{M}'' = 0$  and there is a short exact sequence

$$(3.5) \quad 0 \rightarrow \pi_*(M') \rightarrow \pi_*(M) \rightarrow \pi_*(M'') \rightarrow 0.$$

As remarked in the proof of Theorem 3.2,  $M''$  is weakly equivalent to a wedge of copies of suspensions of the  $A$ -module  $D$ . So  $\pi_*(M'')$  is a direct sum of copies of suspensions of  $\pi_*(D)$ , hence by Lemma 3.3,  $\pi_*(M'') \neq 0$ . The fibre sequence (3.4) induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_*(D \wedge_D M') & \longrightarrow & \pi_*(D \wedge_D M) & \longrightarrow & \pi_*(D \wedge_D M'') \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \pi_*(D \wedge_A M') & \longrightarrow & \pi_*(D \wedge_A M) & \longrightarrow & \pi_*(D \wedge_A M'') \\ & & & & & & \downarrow \\ & & & & & & \pi_*(D \wedge_D M'') \end{array} \quad \begin{array}{c} \curvearrowright \\ = \end{array}$$

in which a non-zero element  $x \in \pi_*(D \wedge_D M'')$  lifts to  $\pi_*(D \wedge_D M)$  and so is in the image of composition passing through  $\pi_*(D \wedge_A M)$ . Therefore  $\pi_*(D \wedge_A M) \neq 0$ .  $\square$

#### 4. LUBIN-TATE COHOMOLOGY OF CLASSIFYING SPACES

We will denote by  $E$  any Lubin-Tate spectrum such as  $E_n$  or  $E_n^{\text{nr}}$ , and then  $K$  will denote the corresponding version of Morava  $K$ -theory see [3] for details. The spectrum  $E$  is a commutative  $S$ -algebra, while  $K$  is an  $E$ -algebra in the sense of [7]. The homotopy groups  $\pi_* E$  and  $\pi_* K$  are 2-periodic and  $\pi_0 E$  is Noetherian;  $\pi_0 K$  is a field, although  $K$  is only homotopy commutative if  $p$  is an odd prime, while when  $p = 2$  it is not even that. Nevertheless, we will view  $K$  as a kind of ‘topological division ring’.

The following lemma will allow us in certain circumstances to relate modules over  $E^{BG} = F(BG_+, E)$  to modules over  $K^{BG} = F(BG_+, K)$ .

**Lemma 4.1.** *For any  $E^{BG}$ -module  $M$ , there is isomorphism of  $K$ -modules*

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

*In particular, there is an isomorphism of  $K$ -modules*

$$K \wedge_{E^{BG}} E \cong K \wedge_{K^{BG}} K.$$

*Proof.* This follows from an obvious generalization of [7, proposition III.3.10]. Since there are isomorphisms of  $E$ -algebras  $K \cong K \wedge_E E$  and  $K^{BG} \cong K \wedge_E E^{BG}$ , for any  $E^{BG}$ -module  $M$ ,

$$\begin{aligned} K \wedge_{E^{BG}} M &\cong K \wedge_E (E \wedge_{E^{BG}} M) \\ &\cong (K \wedge_K K) \wedge_E (E \wedge_{E^{BG}} M) \\ &\cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M). \end{aligned} \quad \square$$

**Remark 4.2.** By a standard argument making use of the Becker-Gottlieb transfer [5], after  $p$ -localization,  $\Sigma^\infty BG_+$  is a retract of  $\Sigma^\infty BG'_+$  where  $G'$  is any  $p$ -Sylow subgroup of  $G$ . In particular, when  $p \nmid |G|$  we have

$$F(BG_+, E) \sim E, \quad F(BG_+, K) \sim K.$$

**Theorem 4.3.** *Let  $G$  be a finite group.*

- (a) *The  $K$ -cohomology  $K^*(BG_+)$  is a finite dimensional  $K^*$ -vector space and the  $E$ -cohomology  $E^*(BG_+)$  is a finitely generated  $E^*$ -module.*
- (b) *If  $K^*(BG_+)$  is concentrated in even degrees, then  $E^*(BG_+)$  is a free  $E^*$ -module of finite rank and*

$$K^*(BG_+) = K^* \otimes_{E^*} E^*(BG_+) = E^*(BG_+)/\mathfrak{m}E^*(BG_+).$$

- (c)  *$K^*(BG_+)$  is an augmented Artinian local  $K^*$ -algebra whose maximal ideal is nilpotent. Hence  $E^*(BG_+)$  is an augmented pro-Artinian local  $E^*$ -algebra,*

$$E^*(BG_+) = \lim_r E^*(BG_+)/\mathfrak{m}^r E^*(BG_+).$$

*Proof.* (a) See [8, 9] for example.

(b) See [10, proposition 2.5].

(c) Following Remark 4.2, we can reduce to the case where  $G$  is a  $p$ -group using the transfer associated with a  $p$ -Sylow subgroup  $G' \leq G$ . The case of a cyclic  $p$ -group  $C_{p^r}$  is well known and

$$K^*(BC_{p^r}_+) = K^*[y]/(y^{p^r}).$$

The case of a general  $p$ -group  $G$  of order  $p^m$  follows by induction on  $m$  since there is always a normal subgroup  $N \triangleleft G$  of index  $p$  and this permits an argument with the Serre spectral sequence associated with the fibration

$$BN \longrightarrow BG \longrightarrow BC_p$$

as used in [16] to calculate  $K^*(BG_+)$  from knowledge of  $K^*(BN_+)$  as input.  $\square$

It is known that  $K^*(BG_+)$  need not be concentrated in even degrees [11].

We are interested in the  $E$ -algebras  $E^{BG} = F(BG_+, E)$  and  $K^{BG} = F(BG_+, K)$ , each of which is  $K$ -local. Of course the diagonal  $BG \longrightarrow BG \times BG$  induces the product on each of these, but only  $E^{BG}$  is strictly commutative, while  $K^{BG}$  is homotopy commutative when  $p \neq 2$  and merely associative when  $p = 2$ . At the level of homotopy groups,  $E^*(BG_+) = \pi_*(E^{BG})$  and  $K^*(BG_+) = \pi_*(K^{BG})$  are both graded commutative.

Now we can apply our earlier results to give

**Theorem 4.4.** *For any finite group  $G$ ,  $E$  and  $K$  are faithful  $E^{BG}$ -modules in the  $K$ -local category.*

*Proof.* It suffices to show that  $K$  is faithful. By Lemma 4.1, for any  $E^{BG}$ -module there is an isomorphism

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

The natural morphism of  $E$ -algebras

$$K \wedge_E F(BG_+, E) \longrightarrow F(BG_+, K \wedge_E E)$$

is a weak equivalence since  $K$  is a finite cell  $E$ -module, so by [7, theorem III.4.2] it is enough to know that

$$(K \wedge_E E) \wedge_{K^{BG}} (K \wedge_E M) \cong K \wedge_{K^{BG}} (K \wedge_E M) \simeq *.$$

If  $M$  is  $K$ -local and non-trivial, then  $K \wedge_{K^{BG}} (K \wedge_E M) \simeq *$ , because we know from Theorem 3.4 that  $K$  is faithful as a  $K^{BG}$ -module.  $\square$

## 5. GALOIS THEORY AND $E^{BG}$

In this section we will consider extensions of the form

$$E^{BG} = F(BG_+, E) \longrightarrow F(EG_+, E) \sim E$$

with  $G$  a finite group and consider whether or not they are Galois. Since we know they are faithful, the issue is whether such an extension satisfies the unramified condition that the map

$$\Theta: F(BG_+, E) \wedge_{E^{BG}} F(BG_+, E) \longrightarrow F(G_+, E)$$

is weak equivalence, and therefore there is a weak equivalence

$$(5.1) \quad E \wedge_{E^{BG}} E \xrightarrow{\sim} \prod_G E.$$

In particular, this condition implies that  $\pi_*(E \wedge_{E^{BG}} E)$  is concentrated in even degrees.

We begin by considering the case of cyclic  $p$ -groups  $C_{p^r}$ .

**Theorem 5.1.** *For each  $r \geq 1$ , the extension*

$$E^{BC_{p^r}} = F(BC_{p^r}_+, E) \longrightarrow F(EC_{p^r}_+, E)$$

*is ramified and hence it is not  $C_{p^r}$ -Galois.*

*Proof.* We recall (see for example [9, lemma 5.1]) that

$$(E^{BC_{p^r}})_* = E^*[[y]]/([p^r]y),$$

where  $y \in (E^{BC_{p^r}})_0 = E^0(BC_{p^r}_+)$  and the  $p$ -series  $[p]y$  has the form

$$[p]y \equiv y^{p^n} \pmod{\mathfrak{m}},$$

so for each  $r \geq 1$  the  $p^r$ -series is inductively defined by

$$\begin{aligned} [p^r]y &= [p]([p^{r-1}]y) = p^r y + \cdots + y^{p^{r^n}} + \cdots \\ &\equiv y^{p^{r^n}} \pmod{\mathfrak{m}}. \end{aligned}$$

By the Weierstrass preparation theorem, there is a polynomial

$$\langle p^r \rangle y = p^r + \cdots + y^{p^{r^n}-1} \equiv y^{p^{r^n}-1} \pmod{\mathfrak{m}}$$

for which

$$[p^r]y = y \langle p^r \rangle y (1 + y f_r(y)),$$

where  $f_r(y) \in E^*[[y]]$ . Then we have

$$(E^{BC_{p^r}})_* = E^*[[y]]/(y \langle p^r \rangle y).$$

The  $(E^{BC_{p^r}})_*$ -module  $E_*$  admits the periodic minimal free resolution

$$(5.2) \quad 0 \leftarrow E_* \leftarrow (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \leftarrow \cdots,$$

so  $\text{Tor}_{*,*}^{(E^{BC_{p^r}})_*}(E_*, E_*)$  is the homology of the complex

$$0 \leftarrow E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I \otimes y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I \otimes \langle p^r \rangle y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \\ \xleftarrow{I \otimes y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I \otimes \langle p^r \rangle y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \longleftarrow \dots,$$

which is equivalent to

$$(5.3) \quad 0 \leftarrow E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \longleftarrow \dots$$

Since  $E_*$  is torsion-free, for  $s \geq 0$  this gives

$$(5.4) \quad \text{Tor}_{s,*}^{(E^{BC_{p^r}})_*}(E_*, E_*) = \begin{cases} E_* & \text{if } s = 0, \\ E_*/p^r E_* & \text{if } s \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus in the Künneth spectral sequence

$$(5.5) \quad E_{s,t}^2 = \text{Tor}_{s,t}^{(E^{BC_{p^r}})_*}(E_*, E_*) \implies \pi_{s+t}(E \wedge_{E^{BC_{p^r}}} E)$$

there can be no non-trivial differentials since for degree reasons the only possibilities involve  $E_*$ -module homomorphisms of the form

$$d^{2k-1}: E_{2k-1,t}^2 = E_t/p^r E_t \longrightarrow E_{0,t+2k-2}^2 = E_{t+2k-2},$$

with torsion-free target. This shows that the odd degree terms in  $\pi_*(E \wedge_{E^{BC_{p^r}}} E)$  are not zero, contradicting the unramified condition 5.1 for a Galois extension.  $\square$

**Remark 5.2.** If we work rationally, then the Künneth spectral sequence

$$E_{s,t}^2(C_{p^r}; \mathbb{Q}) = \text{Tor}_{s,t}^{((E^{BC_{p^r}})\mathbb{Q})_*}(E_*\mathbb{Q}, E_*\mathbb{Q}) \implies \pi_{s+t}(E\mathbb{Q} \wedge_{(E^{BC_{p^r}})\mathbb{Q}} E\mathbb{Q})$$

has  $E_{s,*}^2(C_p^r; \mathbb{Q}) = 0$  except when  $s = 0$ , giving

$$\pi_*(E\mathbb{Q} \wedge_{(E^{BC_{p^r}})\mathbb{Q}} E\mathbb{Q}) = E_*\mathbb{Q} \otimes_{(E^{BC_{p^r}})_*\mathbb{Q}} E_*\mathbb{Q}.$$

This shows that higher filtration terms in the Künneth spectral sequence 5.5 contribute  $p$ -torsion.

Now we extend Theorem 5.1 to arbitrary  $p$ -groups.

**Theorem 5.3.** *Let  $G$  be a non-trivial  $p$ -group. Then the extension*

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

*is not  $G$ -Galois. More precisely, this extension is ramified:*

$$F(EG_+, E) \wedge_{F(BG_+, E)} F(EG_+, E) \approx \prod_G F(EG_+, E).$$

*Proof.* Choose a non-trivial epimorphism  $G \longrightarrow C_p$ ; then for some  $k \geq 1$  there is a factorization

$$(5.6) \quad C_{p^k} \xrightarrow{\quad} G \xrightarrow{\quad} C_p$$

inducing morphisms between the associated Künneth spectral sequences

$$(5.7) \quad E_{**}^r(C_p) \longrightarrow E_{**}^r(G) \longrightarrow E_{**}^r(C_{p^k}).$$

As we saw in the proof of Theorem 5.1, the two outer spectral sequences have trivial differentials. We will analyze the composite morphism  $E_{**}^2(C_p) \longrightarrow E_{**}^2(C_{p^k})$ .

On choosing generators appropriately, the canonical epimorphism  $C_{p^k} \longrightarrow C_p$  induces the  $E_*$ -algebra monomorphism

$$(E^{BC_p})_* = E_*[[y]]/([p]y) \longrightarrow (E^{BC_{p^k}})_* = E_*[[y]]/([p^k]y); \quad y \mapsto [p^{k-1}]y,$$



hence the induced map between the two resolutions of the form (5.2) is

$$\begin{array}{ccccccc}
0 & \longleftarrow & E_* & \longleftarrow & (E^{BC_p})_* & \xleftarrow{y} & (E^{BC_p})_* \xleftarrow{\langle p \rangle y} (E^{BC_p})_* \xleftarrow{y} \dots \\
& & \downarrow = & & \downarrow \rho_0 & & \downarrow \rho_1 & & \downarrow \rho_2 \\
0 & \longleftarrow & E_* & \longleftarrow & (E^{BC_{p^k}})_* & \xleftarrow{y} & (E^{BC_{p^k}})_* \xleftarrow{\langle p^k \rangle y} (E^{BC_{p^k}})_* \xleftarrow{y} \dots
\end{array}$$

where the vertical maps are given by

$$\rho_{2s}: g(y) \mapsto g([p^{k-1}]y), \quad \rho_{2s-1}: h(y) \mapsto h([p^{k-1}]y)\langle p^{k-1} \rangle y.$$

Applying  $E_* \otimes_{(E^{BC_{p^r}})_*} (-)$  to the first and second rows with  $r = 1$  and  $k$  respectively, we obtain a map of chain complexes

$$\begin{array}{ccccccc}
0 & \longleftarrow & E_* & \xleftarrow{0} & E_* & \xleftarrow{p} & E_* \xleftarrow{0} \dots \\
& & \downarrow = \rho'_0 & & \downarrow \rho'_1 = p^{k-1} & & \downarrow \rho'_2 \\
0 & \longleftarrow & E_* & \xleftarrow{0} & E_* & \xleftarrow{p^k} & E_* \xleftarrow{0} \dots
\end{array}$$

where

$$\rho'_{2s} = \text{id}, \quad \rho'_{2s-1} = p^{k-1} \cdot \dots$$

Applying this to the odd degree terms given in (5.4) we see that the induced map

$$E_*/pE_* \xrightarrow{p^{k-1}} E_*/p^k E_*$$

is always a monomorphism. Therefore in (5.7), the first of the induced morphisms

$$E_{**}^2(C_p) \longrightarrow E_{**}^r(G) \longrightarrow E_{**}^r(C_{p^k})$$

is a monomorphism. There can be no higher differentials killing elements in its image because they map to non-trivial elements of  $E_{**}^2(C_{p^k})$  which survive the right hand spectral sequence. This shows that  $E_{**}^\infty(G)$  contains elements of odd degree, and as in the cyclic group case this is incompatible with the unramified condition.  $\square$

We can extend this result to the class of  $p$ -nilpotent groups. A finite group  $G$  is  $p$ -nilpotent if one and hence each  $p$ -Sylow subgroup  $P \leq G$  has a normal  $p$ -complement, *i.e.*, there is a normal subgroup  $N \triangleleft G$  with  $p \nmid |N|$  and  $G = PN = P \rtimes N$ . A convenient summary of the properties of such groups can be found in [14, section 7], see also [18].

**Corollary 5.4.** *If  $G$  is a  $p$ -nilpotent group for which  $p$  divides  $|G|$ , then the extension*

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

*is ramified and so is not  $G$ -Galois.*

*Proof.* By a result of Tate [21],  $G$  being  $p$ -nilpotent is equivalent to the restriction homomorphism giving an isomorphism

$$\text{res}_P^G: H^*(BG; \mathbb{F}_p) \xrightarrow{\cong} H^*(BP; \mathbb{F}_p),$$

and in fact it is sufficient that this holds in degree 1. Comparison of the Serre spectral sequences for  $K^*(BG_+)$  and  $K^*(BP_+)$  shows that

$$K^*(BG_+) \xrightarrow{\cong} K^*(BP_+).$$

It now follows that

$$E^*(BG_+) \xrightarrow{\cong} E^*(BP_+).$$

and the result can be deduced from Theorem 5.3.  $\square$

**Remark 5.5.** The condition of  $G$  being a  $p$ -nilpotent group should not be confused with the condition that the conjugation action of  $G$  on  $\mathbb{F}_p[G]$  is nilpotent. The latter is used in [19, proposition 5.6.3] to ensure convergence of the Eilenberg-Moore spectral sequence and so to prove that for such groups

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p)$$

is a  $G$ -Galois extension. The example of  $G = \Sigma_3$ , the third symmetric group, for the prime  $p = 2$  illustrates this. For each of the Sylow 2-subgroups

$$\{\text{id}, (1, 2)\}, \{\text{id}, (1, 3)\}, \{\text{id}, (2, 3)\}$$

has as normal complement

$$N = \{\text{id}, (1, 2, 3), (1, 3, 2)\},$$

therefore  $\Sigma_3$  is 2-nilpotent. However, the  $\Sigma_3$ -module  $\mathbb{F}_2[\Sigma_3]$  contains the 2-dimensional non-trivial simple submodule

$$V = \{x(1, 2) + y(1, 3) + z(2, 3) : x + y + z = 0\},$$

so by Jordan-Hölder theory every composition series for  $\mathbb{F}_2[\Sigma_3]$  must have this as a composition factor. Hence the action of  $\Sigma_3$  on  $\mathbb{F}_2[\Sigma_3]$  cannot be nilpotent.

## 6. SOME OBSERVATIONS ON THE EILENBERG-MOORE SPECTRAL SEQUENCE

In [19, section 5.6], it is shown that for a finite  $p$ -group  $G$ , the Eilenberg-Moore spectral sequence with

$$(6.1) \quad E_{s,t}^2 = \text{Tor}_{s,t}^{H^*(BG_+; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

converges to  $\pi_*(F(G_+, H\mathbb{F}_p)) = \pi_*(\prod_G \mathbb{F}_p)$ . By comparing it with the Künneth spectral sequence for  $\pi_*(H\mathbb{F}_p \wedge_{F(BG_+, H\mathbb{F}_p)} H\mathbb{F}_p)$ , it is also shown that

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p)$$

is a  $G$ -Galois extension.

Let us consider in detail the case  $G = C_p$  for  $p$  an odd prime. The case when  $p = 2$  is similar. First we write

$$H^*(BC_p) = H^*(BC_{p+}; \mathbb{F}_p) = \mathbb{F}_p[y] \otimes \Lambda(z),$$

where  $y \in H^2(BC_p)$  and  $z \in H^1(BC_p)$ . Then (6.1) becomes

$$E_{**}^2 = \Gamma(\sigma z) \otimes \Lambda(\sigma y),$$

where  $\sigma y \in E_{1,-2}^2$  and  $\sigma z \in E_{1,-1}^2$  are the suspensions of  $y$  and  $z$ , see [17]. Writing  $\gamma_r = \gamma_r(\sigma z)$ . The first non-trivial differential is

$$d^{p-1} \gamma_p = \sigma y,$$

and we have

$$E_{**}^p = \mathbb{F}_p[\zeta]/(\zeta^p) \otimes \Gamma(\gamma_{p^2}) \otimes \Lambda(\gamma_p \sigma y),$$

where  $\zeta$  represents the class of  $\sigma z$ . The remaining differentials are determined by the formulae

$$d^{p^s - p^{s-1} - 1} \gamma_{p^s} = \gamma_{p^{s-1}} \sigma y$$

in

$$E_{**}^{p^s - p^{s-1} - 1} = \mathbb{F}_p[\zeta]/(\zeta^p) \otimes \Gamma(\gamma_{p^s}) \otimes \Lambda(\gamma_{p^{s-1}} \sigma y).$$

Finally we have

$$E_{**}^\infty = \mathbb{F}_p[\zeta]/(\zeta^p),$$

which is an avatar of  $\prod_{C_p} \mathbb{F}_p$ . These differentials are forced by the known answer and multiplicativity, and are also related to the discussion of [17, section 6]. For Lubin-Tate theory  $(E^{BC_{p^r}})_*$  is free over  $E_*$  and the comparison of the Eilenberg-Moore with the Künneth spectral sequence together with our Theorems 5.1 and 5.3 has the following consequence.

**Proposition 6.1.** *For the cyclic  $p$ -group  $C_{p^r}$  the  $E$ -theory Eilenberg-Moore spectral sequence for  $BC_{p^r}$  with*

$${}^{\mathrm{L-T}}E_{s,t}^2 = \mathrm{Tor}^{(E^{BC_{p^r}})^*}(E_*, E_*)$$

*does not converge to  $\pi_*(\prod_{C_{p^r}} E)$ .*

Just as in the  $H\mathbb{F}_p$  case, we can compare the Morava  $K$ -theory based Eilenberg-Moore spectral sequence with the Künneth spectral sequence. Work of Bauer [4] on the convergence of the Cotor-version of this Eilenberg-Moore spectral sequence shows that the corresponding spectral sequence converges for  $G = C_p$  and odd primes  $p$ , and therefore

$$K \wedge_{K^{BC_p}} K \sim \prod_{C_p} K.$$

The extension of  $S$ -algebras  $K^{BC_p} \rightarrow K^{EC_p}$  can be interpreted as a Galois extension of non-commutative  $S$ -algebras.

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SCHOOL OF MATHEMATICS & STATISTICS, UNIVERSITY OF GLASGOW, GLASGOW G12 8QW, SCOTLAND.

*E-mail address:* [a.baker@maths.gla.ac.uk](mailto:a.baker@maths.gla.ac.uk)

*URL:* <http://www.maths.gla.ac.uk/~ajb>

FACHBEREICH MATHEMATIK DER UNIVERSITÄT HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY.

*E-mail address:* [birgit.richter@uni-hamburg.de](mailto:birgit.richter@uni-hamburg.de)

*URL:* <http://www.math.uni-hamburg.de/home/richter/>